Tutorial Lecture

Delta-F Particle-in-Cell Plasma Simulation Model

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Outline

• Introduction – Collisionless Plasma Dynamics – Vlasov-Maxwell Equations
Low Noise Particle-in-Cell Approaches

• $\delta F$-PIC Model and Equations

• $\delta F$-PIC Algorithm – Computational Cycle, Code Implementation

• Model Tests – Linear and Nonlinear Landau Damping, Plasma Echo, Beam Instabilities

• Extensions – Drift-Kinetics
Introduction
Why Plasma Simulation?

- Plasma simulation is a tool for studying nonlinear processes, which is mainly the regime in which measurements and observations are made.

- If done correctly, it provides information on the dominant physical processes.

- It can be the basis for the formulation of new theory and provide us with predictive capability.
Kinetic Plasma Model: Vlasov-Maxwell

Vlasov equation in continuity form

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{z}} \cdot (\dot{\vec{z}} f) = 0 \]

where \( \vec{z} = (\vec{x}, \vec{v}) \) and \( \dot{\vec{z}} = (\vec{v}, (q/m)[\vec{E} + \frac{\vec{v} \times \vec{B}}{c}]) \)

Maxwell equations

\[ \nabla \times \vec{E}_T = -\frac{1}{c} \frac{\partial \vec{B}_T}{\partial t} \]
\[ \nabla \times \vec{B}_T = \frac{1}{c} \frac{\partial \vec{E}_T}{\partial t} + \frac{4\pi}{c} \vec{J}_T \]
\[ \nabla \cdot \vec{E}_L = 4\pi \rho \]

where \( \rho = \int q f d^3v \) and \( \vec{J} = \int q f \vec{v} d^3v \)
Continuum vs. Discrete Representation

Methods for solving the Vlasov equation – collisionless plasma dynamics

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{z}} \cdot (\vec{z} f) = 0 \]

Continuum form

\[ f(\vec{z}, t) = \lim_{\Delta \vec{z}, \Delta t \rightarrow 0} f(\Delta \vec{z}, \Delta t) \]

Discrete form

\[ F(\vec{z}, t) = \sum_i \delta(\vec{z} - \vec{z}_i) \]

Accuracy and convergence are important as well as satisfying conservation relations.
Finite-Sized Particle-in-Cell

Equations of motion

$$\ddot{z}_i = (\ddot{v}_i, (q/m) \int d^n x S(\vec{x} - \vec{x}_i)[\vec{E} + \frac{\vec{v}_i \times \vec{B}}{c}])$$

Maxwell equations

$$\nabla \times \vec{E}^T = -\frac{1}{c} \frac{\partial \vec{B}^T}{\partial t}$$

$$\nabla \times \vec{B}^T = \frac{1}{c} \frac{\partial \vec{E}^T}{\partial t} + \frac{4\pi}{c} \vec{J}^T$$

$$\nabla \cdot \vec{E}^L = 4\pi \rho$$

where

$$\rho = \sum_{i=1}^{N} q_i S(\vec{x} - \vec{x}_i) \text{ and } J = \sum_{i=1}^{N} q_i \vec{v}_i S(\vec{x} - \vec{x}_i)$$

References:

Birdsall, Langdon, ’85
Hockney, Eastwood, ’88
Dawson, Rev. Mod. Phys., ’83
Discreteness Effects

Collisionless plasmas require

\[ \frac{\nu}{\omega_p} \approx \frac{1}{n\lambda_D^3} \ll 1 \]

where \( \lambda_D = (T/m)^{1/2}/\omega_p \) and \( \omega_p = (4\pi ne^2/m_e)^{1/2} \)

Number of particles in the Debye sphere characterizes the ‘graininess’ so we define

\[ g \equiv \frac{1}{n\lambda_D^3} \]

and the Vlasov equation is derived in the limit \( g \rightarrow 0 \)

Typically in space plasmas \( g < 10^{-9} \)
Discreteness Effects

PIC typically uses a mesh with $\Delta \sim \lambda_D$

and we can therefore obtain a scaling relation between the number of macro-particles and $g$ as follows

$$N_{part} = nL^d = n(\Delta)^d\left(\frac{L}{\Delta}\right)^d$$

$$= g_{pic}^{-1}\left(\frac{L}{\Delta}\right)^d$$

where $g_{pic} \equiv \frac{1}{n\lambda_D^d}$ and $d = 1, 2$ or $3$

Typically $g_{pic} \sim 10^{-2} - 10^{-6}$

but we can still clearly observe the collective behavior.
Thermal Fluctuations

• Since each macro-particle has a much larger kinetic energy than a real charged particle in space plasmas, we get enhanced thermal fluctuations of the electrostatic and electromagnetic fields.

• For PIC we need to ensure our thermal fluctuation level is much lower that the saturated state of the instabilities we are studying.
Thermal Fluctuations and Discreteness

Very roughly, by comparing the electrostatic field energy density

\[ \left( \sim \frac{T}{\lambda_D} \right) \]

where \( T \equiv m\nu_{th}^2 \)

to the thermal energy density \( \left( \sim nT \right) \) we obtain (in 1D)

\[
\frac{\text{Electrostatic Field En. Density}}{\text{Thermal En. Density}} \approx \frac{T/\lambda_D}{nT} = \frac{1}{n\lambda_D}
\]

\[
= \frac{L_x}{N_{part}} \frac{\Delta x}{\lambda_D} \approx \frac{1}{\text{# particles per cell}}
\]

For \( \delta F\text{-PIC} \) this estimate is modified to be

\[
\frac{L_x}{N_{part}} \frac{\Delta x}{\lambda_D} \left| \frac{\delta f}{f} \right| 2
\]
Low Noise Approaches in PIC

• Quiet Start Method

  - based on ordered phase space loading with minimal correlations

  \[ F(\vec{z}, t) = \sum_i \delta(\vec{z} - \vec{z}_i) \]
  \[ \text{where } \vec{z} = (\vec{x}, \vec{v}) \]

• \( \delta F \)-PIC Method

  \[ f(z, t) = f_0(z) + \delta f(z, t) \text{ gives} \]
  \[ \frac{d\delta f}{dt} = -\frac{df_0}{dt} \]
$\delta F$-PIC Model
Development of $\delta F$-PIC Model

• Early work on linearized particle codes (70’s)
  • Byers, Cohen, Friedberg, et al.

• First versions of nonlinear $\delta f$-PIC model (80’s)
  • Tajima, Perkins (’83)
  • Kotschenruether (’88), Dimits, Lee (’88)

• Fully nonlinear $\delta f$ PIC algorithm (90’s)
  • Parker, Lee (’93)
  • Aydemir (’94), Hu, Krommes (’94)

• Extensions
  • Coupling to MHD (Toda et al, ’99, Parker, Park et al, 2000)
  • Sources (Candy, ’96)
  • Electromagnetic PIC (Sydora, ’99)

• Addition of collisions
  • Xu, Dimits, Cohen (’93)
  • Lin, Lee, Parker, Chen et al. (’95, ’99)
  • Wang et al. (2000)
Splitting the distribution

\[ f(\tilde{z}, t) = f_o(\tilde{z}) + \delta f(\tilde{z}, t) \]

gives

\[ \frac{d\delta f}{dt} = -\frac{df_o}{dt} \]

\[ \dot{z}_o = (\vec{v}, (q/m)[\vec{E}_o + \frac{\vec{v} \times \vec{B}_o}{c}]) \]

\[ \dot{z}_1 = (\vec{v}, (q/m)[\vec{E}_1 + \frac{\vec{v} \times \vec{B}_1}{c}]) \]

Evolution equation for \( \delta f \)

\[ \frac{\partial \delta f}{\partial t} + \frac{\partial}{\partial \tilde{z}} \cdot (\dot{\tilde{z}} \delta f) = -\dot{z}_1 \cdot \frac{\partial f_o}{\partial \tilde{z}} \]

nonlinear trajectory: \( \tilde{z} = \tilde{z}_o + \tilde{z}_1 \)
Delta-F Method

Representation of $\delta f$

$$\delta f(\ddot{z}, t) = \sum_i w_i \delta(\ddot{z} - \ddot{z}_i)$$

where the particle weight is

$$w_i = \frac{\delta f}{g}$$

and $g$ is an arbitrary 'marker' distribution

$$g(\ddot{z}, t) = \sum_i \delta(\ddot{z} - \ddot{z}_i)$$
Delta-F Evolution Equation

\[ \delta f \text{ evolution equation} \]

\[ \frac{\partial \delta f}{\partial t} + \frac{\partial}{\partial \tilde{z}} \cdot (\dot{\tilde{z}} \delta f) = -\dot{\tilde{z}}_1 \cdot \frac{\partial f_o}{\partial \tilde{z}} \]

becomes

\[ \frac{dw_i}{dt} = -\dot{\tilde{z}}_1 \cdot \frac{1}{g(\tilde{z},t)} \frac{\partial f_o}{\partial \tilde{z}} \bigg|_{\tilde{z}} \]

and since both \( f \) and \( g \) satisfy \( df/dt=0 \) and \( dg/dt=0 \)

\[ \frac{dw_i}{dt} = -\left( \frac{f(0)}{g(0)} - w_i \right) \dot{\tilde{z}}_1 \cdot \frac{1}{f_o(\tilde{z},t)} \frac{\partial f_o}{\partial \tilde{z}} \bigg|_{\tilde{z}} \]

Division by \( g \) is, however, “mathematically ugly”
\[ \delta F - \text{PIC Model Equations} \]

An alternative derivation

Let \( \vec{F} = \frac{q}{m}(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}) \) and consider the evolution of the total distribution, \( f(\vec{x}, \vec{v}, t) \) given as

\[ \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \vec{F} \cdot \frac{\partial f}{\partial \vec{v}} = 0 \]

The total distribution function evolves along the characteristics

\[ \frac{d\vec{x}}{dt} = \vec{v} \]
\[ \frac{d\vec{v}}{dt} = \vec{F} \]

which are also the particle trajectories, such that

\[ \frac{df}{dt} = 0 \]

where \( \frac{d}{dt} \) is the total derivative evaluated along the characteristics.
We may now split the distribution function and the force into background and perturbation pieces

\[ f = f_o + \delta f \]
\[ \vec{F} = \vec{F}_o + \vec{\delta F} \]

and upon substitution into \( df/dt = 0 \) leads to the equation

\[ \frac{\partial \delta f}{\partial t} + \vec{v} \cdot \frac{\partial \delta f}{\partial \vec{x}} + (\vec{F}_o + \delta \vec{F}) \cdot \frac{\partial \delta f}{\partial \vec{v}} = -\left[ \frac{\partial f_o}{\partial t} + \vec{v} \cdot \frac{\partial f_o}{\partial \vec{x}} + (\vec{F}_o + \delta \vec{F}) \cdot \frac{\partial f_o}{\partial \vec{v}} \right] \]

\[ = f_o \left[ \frac{\partial \ln(f_o)}{\partial t} + \vec{v} \cdot \frac{\partial (\ln(f_o))}{\partial \vec{x}} + (\vec{F}_o + \delta \vec{F}) \cdot \frac{\partial \ln(f_o)}{\partial \vec{v}} \right] \]

\[ = f_o \frac{d\ln f_o}{dt} \]

and therefore

\[ \frac{d\delta f}{dt} = -f_o \frac{d\ln(f_o)}{dt} \]
δF-PIC Model Equations

Now a dual set of weights is introduced such that

\[ f_o = pg \]
\[ \delta f = wg \]

where \( p \) and \( w \) are marker weights for each piece of the distribution function and \( g \) represents the marker distribution function. From the total \( f \) evolution equation

\[
\frac{df}{dt} = \frac{d(f_o + \delta f)}{dt} = \frac{d(pg + wg)}{dt}
\]

\[ = g\left(\frac{dp}{dt} + \frac{dw}{dt}\right) = 0 \]

and

\[
\frac{dp}{dt} = -\frac{dw}{dt}
\]

which also leads to the constant of motion \( p(t) + w(t) = \text{constant} \).
The nonlinear characteristics for the $\delta f$ evolution are

\[
\begin{align*}
\frac{d\vec{x}}{dt} &= \vec{v} \\
\frac{d\vec{v}}{dt} &= \vec{F}_o + \delta \vec{F} \\
\frac{dw}{dt} &= -p \frac{dln(f_o)}{dt} \\
\frac{dp}{dt} &= p \frac{dln(f_o)}{dt}
\end{align*}
\]

The last 2 equations can be combined into a single weight evolution equation: we use the condition $w(0)=0$ (or $\delta f(t = 0) = 0$) and for a marker distribution matching $f_o$ (or $g = f_o$) we have $p(0)=1$. This gives

\[p(t) + w(t) = constant = p(0) + w(0) = 1\]

leading to the combined weight evolution

\[
\frac{dw}{dt} = -(1 - w) \frac{dln(f_o)}{dt}
\]
\( \delta F \)-PIC Model Equations

For the electrostatic case, considered in this work, the force consists of the external and self-consistent electric fields

\[
\vec{F}_o + \delta \vec{F} = \frac{q}{m} (\vec{E}_o + \delta \vec{E})
\]

where \( \delta \vec{E} \) is obtained from

\[
\vec{\nabla} \cdot \delta \vec{E} = 4\pi \Sigma_s \tilde{\rho}_s
\]

and \( \Sigma_s \) is a sum over the charged particle species, \( s \), and \( \tilde{\rho}_s = q_s \int \delta f d\vec{v} \).

The perturbed charge density is obtained from the \( \delta f \) weights, \( w_i \), using

\[
\tilde{\rho} = \sum_i q_i w_i S(\vec{x} - \vec{x}_i)
\]
Conservation Properties

In the absence of particle sources and sinks the particle number should be conserved and the zeroth velocity moment gives

\[ \frac{\partial}{\partial t} \int \delta f \, d\vec{x} = 0 \]

or

\[ \sum_{i=1}^{N} w_i(t) = 0 \]

if the sum of the weights is nearly zero initially. The first velocity moment leads to total momentum conservation

\[ \sum_{\alpha} m_{\alpha} \sum_{i=1}^{N} v_{\alpha i} w_{\alpha i}(t) = 0 \]

where \( \alpha = \) electron and ion species. The second velocity moment gives the total energy conservation relation

\[ \sum_{\alpha} \frac{m_{\alpha}}{2} \sum_{i=1}^{N} v_{\alpha i}^2 w_{\alpha i}(t) + \frac{1}{8\pi} \int |\delta \vec{E}|^2 \, d\vec{x} = 0 \]
Thermal Equilibrium Properties

A plasma sustains fluctuations of various modes of oscillation in thermal equilibrium. For the case of unmagnetized electrostatic oscillations in a collisionless plasma, the full-f PIC simulation model gives an electric field energy equipartition per wavenumber as

\[
< \frac{E_k^2 L}{8\pi} > = \frac{T/2}{1 + k^2 \lambda_D^2 S^2(k)}
\]

where \( S(k) \) is the finite-sized particle shape factor, taken to be Gaussian, \( S(k) = e^{k^2 a^2 / 2} \).

For the \( \delta f \)-PIC model the fluctuation-dissipation theory gives a similar equipartition

\[
< \frac{E_k^2 L}{8\pi} > = \frac{(T/2) \bar{w}^2}{1 + k^2 \lambda_D^2 S^2(k)}
\]

where \( \bar{w}^2 \) is the spatially averaged \( |\delta f / f|^2 \).
Marker Particle Diffusion

In the $\delta f$-PIC method the weight, $w$, of the marker particles obeys an evolution equation of the form

$$\frac{dw_i}{dt} = R_i(t)$$

where

$$R_i(t) = \vec{v}_i \cdot \frac{\partial}{\partial \vec{x}_i} \ln(f_o) + \vec{F}_i \cdot \frac{\partial}{\partial \vec{v}_i} \ln(f_o)$$

This equation can be formally integrated to give

$$w_i(t) = \int_0^t dt' R_i(t')$$

This has a time evolution determined by the two-time correlation function

$$\langle w_i^2(t) \rangle = \int_0^t dt' \int_0^t dt'' < R_i(t')R_i(t'') >$$
δF-PIC Algorithm
\( \delta F \)-PIC Computational Cycle

1. \( \Delta t \)
2. Field Interpolation and particle push:
   \[ (\vec{E} = \vec{E}_o + \vec{E}_1, \quad \vec{B} = \vec{B}_o + \vec{B}_1) \]
3. Particle Interpolation:
   \[ \rho_1 = \sum_i q_i w_i S(\vec{x} - \vec{x}_i) \]
4. Field solve:
   \[ (\rho_1, \vec{J}_1) \]
5. Initialize positions, velocities, weights:
   \[ (\vec{z}_i, w_i) \]
Numerical Algorithm Implementation

Characteristics – integrate using Leapfrog algorithm

\[
\begin{align*}
\dot{\vec{z}}_0 &= (\vec{v}, (q/m)[\vec{E}_0 + \frac{\vec{v} \times \vec{B}_0}{c}]) \\
\dot{\vec{z}}_1 &= (\vec{v}, (q/m)[\vec{E}_1 + \frac{\vec{v} \times \vec{B}_1}{c}])
\end{align*}
\]

Weight evolution

\[
\frac{dw_i}{dt} = -(1 - w_i) \vec{z}_1 \cdot \frac{1}{f_o(\vec{z})} \frac{\partial f_o}{\partial \vec{z}} |\vec{z}
\]

nonlinear trajectory: \( \vec{z} = \vec{z}_0 + \vec{z}_1 \)
Numerical Algorithm Implementation

Nonlinear Characteristics

\[
\begin{align*}
\dot{z}_0 &= (\vec{v}, \frac{q}{m})[\vec{E}_0 + \vec{v} \times \vec{B}_0] \\
\dot{z}_1 &= (\vec{v}, \frac{q}{m})[\vec{E}_1 + \frac{\vec{v} \times \vec{B}_1}{c}]
\end{align*}
\]

Integrate using second order accurate Leapfrog scheme (Buneman)

\[
\begin{align*}
\bar{x}_i^{n+1/2} &= \bar{x}_i^{n-1/2} + \Delta t \bar{v}_i^n \\
\bar{v}_i^{n+1} &= \bar{v}_i^n + \Delta t \bar{F}_i^{n+1/2}
\end{align*}
\]
Numerical Algorithm Implementation

Characteristics – integrate using Leapfrog algorithm

\[
\begin{align*}
\dot{\mathbf{z}}_o &= (\mathbf{v}, (q/m) [\mathbf{E}_o + \frac{\mathbf{v} \times \mathbf{B}_o}{c}]) \\
\dot{\mathbf{z}}_1 &= (\mathbf{v}, (q/m) [\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c}])
\end{align*}
\]

Weight evolution – centered time difference

\[
\frac{dw_i}{dt} = -(1 - w_i) \dot{\mathbf{z}}_1 \cdot \left| \frac{1}{f_o(z)} \frac{\partial f_o}{\partial z} | z \right|
\]

\[
\downarrow
\]

\[
w_i^{n+1} = w_i^n - \Delta t(1 - w_i^n) \dot{\mathbf{z}}_1^{n+1/2} \cdot \left( \frac{1}{f_o(z)} \frac{\partial f_o}{\partial z} \right)^{n+1/2}
\]
Technical note: If we want to use the leapfrog time integration method we have to be careful in setting up the right hand side of the weight evolution.

Predictor-corrector is an alternative approach, using integer time levels.
Initially: $\tilde{x}_{i}^{m-1/2}, \tilde{v}_{i}^{m}, w_{i}^{n}, w_{i}^{n-1}$ are given.

Step 1: Advance $\tilde{x}_{i}^{m-1/2}$ to $\tilde{x}_{i}^{m+1/2}$ using $\tilde{x}_{i}^{m+1/2} = \tilde{x}_{i}^{m-1/2} + \Delta t \tilde{v}_{i}^{m}$.
Determine $w_{i}^{n+1/2}$ using $w_{i}^{n+1/2} = (3/2)w_{i}^{n} - (1/2)w_{i}^{n-1}$.

Step 2: Accumulate the perturbed charge density on the grid using
\[
\rho^{n+1/2}(\tilde{x}) = \sum_{i=1}^{N} q_{i} w_{i}^{n+1/2} S(\tilde{x} - \tilde{x}_{i}^{m+1/2}).
\]

Step 3: Transform $\rho^{n+1/2}$ to k-space giving $\rho^{n+1/2}(\tilde{k})$.
Solve for $\tilde{E}^{n+1/2}(\tilde{k})$ using $\rho^{n+1/2}(\tilde{k})$.

Step 4: Transform $\tilde{E}^{n+1/2}(\tilde{k})$ to real space.

Step 5: Advance the velocities to $\tilde{v}_{i}^{n+1}$ using the interpolated electric field at the particle and $\tilde{v}_{i}^{n+1} = \tilde{v}_{i}^{n} + \Delta t \tilde{E}_{i}^{n+1/2}(\tilde{x}^{n+1/2})$.

Step 6: Advance the weights to $w_{i}^{n+1}$ using $v_{i}^{n+1/2} = (v_{i}^{n+1} + v_{i}^{n})/2$ in the source term.

Step 7: Return to Step 1.
Particle Loading

c… load the particles uniformly in x and Gaussian in velocity
c… with a thermal velocity width, vtx. Initialize weights, w_i

\[
\text{do } i = 1, \text{ no} \\
\quad x(i) = (i-1) \times \Delta x + \text{ancxl} \\
\quad v_x(i) = (\text{vtx} \times \text{ranorm}(d)) + \text{vdx} \\
\quad w(i) = v_{\text{eps}} \times \cos(ako \times x(i)) \\
\quad \text{wold}(i) = w(i) \\
\text{enddo}
\]
Code Implementation

**Charge Density**

The perturbed charge density is obtained from the $\delta f$ weights, $w_i$, using

$$\rho_1 = \sum_i q_i w_i S(\vec{x} - \vec{x_i})$$

c...electron charge density

```fortran
   do  i = 1, nox
      lx  =  x(i) + 1.5
      dx  = (x(i) - lx +1.)*0.5
      lxl = il(lx)
      lxr = ir(lx)

      rhoe(lx)  = rhoe(lx)  + w(i)
      rhoe(lxl) = rhoe(lxl) – dx*w(i)
      rhoe(lxr) = rhoe(lxr) + dx*w(i)
   enddo
```
Code Implementation

**Field Solver – k-space**

$\delta \vec{E}$ is obtained from $\vec{\nabla} \cdot \delta \vec{E} = 4\pi \sum_s \rho_{1s}$ where $s$ is the species.

\[
\delta \vec{E}(x) = \sum_k \delta E_k e^{ikx}
\]

\[
\rho_1(x) = \sum_k \rho_{1k} e^{ikx}
\]

\[
\delta E_k = \frac{4\pi \rho_{1k}}{ik}
\]
c.....transform charge to k-space

    call fft2(rho,-1)

c.....calculation of electric field in k-space

    do  i=1, ncx / 2

        i2   = i  + i
        i1   = i2 - 1

        elx(i1) = fex(i)  * rho(i2)                /*    (real part)
        elx(i2) =-fex(i)  * rho(i1)                /*   (imaginary part)

    enddo

c.....transform electric field back to real space

    call fft2(elx,1)
Particle Push

do i = 1, nox

lx = x(i) + 1.5
dx = ( x(i) - lx + 1. )*0.5
lxl = il(lx)
lxr = ir(lx)

ext = elx(lx) + dx * ( elx(lxr) - elx(lxl) ) /* interpolate field to particle location

vxold = vx(i)

vx(i) = vx(i) + (bete *ext)

vxnew = 0.5 * ( vxold + vx(i) )

wold(i) = w(i)

w(i) = w(i) + ( (1.-w(i)) * (vxnew/vtx**2) *ext /* update weight

enddo
Model Tests
Full-F PIC Simulations- Landau Damping
$\delta f = (1 + \epsilon \cos(kx)) f_0$

$\gamma_L \approx -\left(\frac{\pi}{8}\right)^{1/2} \frac{1}{(k\lambda_D)^3} \exp\left[-\frac{1}{2k^2\lambda_D^2} - \frac{3}{2}\right]$
δf-PIC Simulations- Trapping Oscillations

Bounce period: \( \tau_B \approx (2m_e/e k^2 |\phi_k|)^{1/2} \)

Wave amplitude threshold: \( |\phi_k| \approx \frac{m_e}{ek^2} \gamma_L(k)^2 \)

\[ \delta f = (1 + \epsilon \cos(kx)) f_o \]
BGK Modes

The two counterpropagating BGK modes can be obtained from the stationary solution to the Vlasov-Poisson system and by applying two Galilean transformations with velocities $V$ and $-V$ which gives distributions $f_1(x - Vt, v - V)$ and $f_2(x + Vt, v + V)$ and electric fields $E(x - Vt)$ and $E(x + Vt)$. The single particle energies in the two moving frames are $K_{1,2} = (v \pm V)^2 / 2 - \phi(x \pm Vt)$ and the distributions can be expressed in terms of these energies as $f_1(x - Vt, v - V) = g(K_1)$ and $f_2(x + Vt, v + V) = g(K_2)$. The electric potential divides the electrons into three populations consisting of trapped particles, untrapped particles with velocities larger than the BGK phase velocity $V$ and untrapped particles smaller than the BGK phase velocity.
δf-PIC Simulations - Trapped Particle
BGK Equilibrium

Electrostatic Energy (Arb. Units)

Time ($\omega_{pe} t$)
Plasma Echo (Temporal)

For the temporal plasma echo simulation we consider a 1D simulation configuration, unmagnetized, with electrons only; the ions are considered a charge neutralizing background. We excite two pulses using an external electric field of the form

\[ E(x,t) = E_1 \cos(k_1 x) \delta(t) + E_2 \cos(k_2 x) \delta(t - \tau) \]  \hspace{1cm} (1)

The two pulses both excite density modulations which exponentially decay at approximately the Landau damping rate. After a time longer than the inverse Landau damping rate of the first two pulses, a third echo wave appears as a density modulation with wavenumber \( k_{echo} = k_2 - k_1 \). The echo time is given as (O’Neil and Gould, 1968)

\[ \tau_{echo} = \frac{k_2}{k_2 - k_1} \tau \]  \hspace{1cm} (2)

For the \( \delta f \)-PIC simulation we chose physical parameters: \( k_1 \lambda_D = 0.483, k_2 \lambda_D = 0.966, \tau = 20 \omega^{-1}_p \), and \( E_1 = E_2 = 0.18 \) where E is normalized as \( \epsilon E/m_e \omega_p^2 \Delta \) and \( \Delta = 1.64 \lambda_D \). N=2560000 particles were used.
Non-Equilibrium Distributions

Beam-type distributions

\[ f_o = (1-\epsilon)(m_e/2\pi T_e)^{3/2}e^{-m ev^2/2T_e} + \epsilon(m_e/2\pi T_b)^{3/2}e^{-m e(v-v_b)^2/2T_b} \]

where \( \epsilon = n_b/n_o \)

Anisotropic thermal

\[ f_o = (m_e/2\pi T_\perp)(m_e/2\pi T_\parallel)^{1/2}e^{-m e v_\perp^2/2T_\perp - m e v_\parallel^2/2T_\parallel} \]
$\delta f$-PIC Simulations - Beam Instability

$$\gamma_L \approx -\left(\frac{\pi}{8}\right)^{1/2} \left(\frac{m_e}{T_e}\right)^{3/2} \frac{\omega_r^4}{k^3} (1 - \epsilon) \exp\left[-\frac{m_e \omega_r^2}{2T_e k^2}\right] + \epsilon \left(\frac{T_e}{T_b}\right)^{3/2} \left(\frac{k v_b - \omega_r}{\omega_r}\right) \exp\left[-\frac{m_e (\omega_r - k v_b)^2}{2T_b}ight]$$

$n_b/n_\circ = 0.001$
$T_b/T_e = 1, \nu_{De} = 8 \nu_{Te}, n_b/n_o = 3 \times 10^{-3}$

$T_b/T_e = 0.36, \nu_{De} = 10 \nu_{Te}, n_b/n_o = 3 \times 10^{-3}$
$T_b/T_e = 1, \nu_D e = 8\nu_T e, n_b/n_o = 3 \times 10^{-3}$
\[ \frac{T_b}{T_e} = 0.36, \ v_{De} = 10v_{Te}, \ \frac{n_b}{n_o} = 3 \times 10^{-3} \]
Coherent wave packet observations

Geotail Observations (Usui et al., JGR, 2005)

Figure 5. Estimation of temporal variation of $v_p$ by using the Geotail data of 13 January 1994. (a) Waveform of Langmuir wave observed by the Geotail spacecraft at (GSM-X, GSM-Y, GSM-Z) = (91, 18, 4[R_E]) on 13 January 1994. (b) Its temporal packets extracted from the waveform envelope. (c) Estimation of temporal variation of $v_p$ in a solid line.
δF-PIC Model
Extensions
Other Vlasov Kinetic Models

\[ f(z, t) = f_o(z) + \delta f(z, t) \] gives

\[ \frac{d \delta f}{dt} = -\frac{df_o}{dt} \]

Full Magnetized Particle Dynamics

\[ \frac{d \delta f}{dt} = -[\vec{v} \cdot \nabla f_o + \frac{q}{m} (\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}) \cdot \nabla_v f_o]|_{(\vec{x}_j, \vec{v}_j)} \]

Drift Dynamics (Drift-kinetics)

\[ \frac{d \delta F}{dt} = -[(v_\parallel \vec{b} + v_D) \cdot \nabla F_o + \frac{q}{m} \vec{E} \cdot \nabla_v F_o]|_{(\vec{R}_j, v_\parallel j)} \]

\[ F = \sum_i^N \delta(\vec{R} - \vec{R}_i) \delta(v_\parallel - v_\parallel i) \delta(\mu - \mu_i) \]

\[ \mu_i = \frac{v_{ij}}{2B} \]
Comparison – Full F versus $\delta F$

(Low frequency kinetic tearing instability)

(Naitou, Lee, RS, 1995)
Final Points

- The $\delta F$-PIC method can be used to do linear and nonlinear PIC simulations.

- Nonlinear $\delta F$-PIC most useful when $\delta f/f$ is very small.

- The method can be adapted to work in standard PIC codes to obtain low noise simulations such as electrostatic, Darwin, relativistic electromagnetic or hybrid.

- Some pitfalls include difficulty in obtaining a precise energy conservation and there is the issue of marker diffusion that can be a problem for very long time scale simulations.